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We consider a modified “Swiss cheese” model in Brans-Dicke theory, and use it to discuss the evolution of black holes in an expanding universe. We define the black hole radius by the Misner-Sharp mass and find their exact time evolutions for dust and vacuum universes of all curvatures.

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I. INTRODUCTION

The evolution of primordial black holes in scalar-tensor theories has been studied in the literature by several authors [1–6]. Some questions about what happens to a black hole in an expanding universe when G varies in spacetime in these theories were posed by Barrow [1], who considered two possible scenarios: (a) the effective gravitational “constant”, $G(t)$, at the black hole horizon changes along with its cosmological evolution so that the size of a black hole is approximated by $R = 2G(t)M$. (b) G remains constant at the black hole event horizon while it evolves on larger scales; a large inhomogeneity in G is therefore generated. The case (b) was called “gravitational memory” because the black hole remembers the value of G at its formation time. In either case the observational constraints on the abundance of exploding primordial black holes deduced from the total radiation backgrounds today would be modified [2].

Scheel, Shapiro and Teukolsky [3] made numerical analyses of dust collapse in Brans-Dicke (BD) theory, showing that the surface area of the event horizon decreases with time, contrary to the case in Einstein theory. Kang [4] gave an analytic explanation for the surface-area decrease in BD theory. Later, Jacobson claimed that there is no “gravitational memory” effect, by analyzing the evolution of a scalar field $\phi(t, r)$ in Schwarzschild background [5]. He found a particular solution of the scalar wave equation which matches smoothly between the black hole and a special cosmological background and showed that ϕ at the event horizon evolves along with its asymptotic value $\phi(t, \infty)$. However, this solution requires a particular cosmological variation of $G(t)$ to occur in the background and may be special. It was also argued that even if the black hole mass in the Einstein frame is constant then its mass in the Jordan frame is time-dependent.

In order to investigate the time-dependence of the black-hole mass, Saida and Soda constructed a “cell lattice” universe in BD theory [6]. In their model the universe is first tessellated by identical polyhedrons, which are then replaced by Schwarzschild black holes. It was shown that the black-hole mass has an adiabatic time dependence, which is qualitatively different according to the sign of the curvature of the background universe.

As an extension of Saida and Soda’s work, we consider a “Swiss cheese” (or Einstein-Straus) model [7] in BD theory and discuss the evolution of the radius and mass of black holes in an expanding isotropic background universe. The usual “Swiss cheese” model refers to a cosmological model in which spherical regions in the Friedmann-Robertson-Walker (FRW) universe are replaced by Schwarzschild spacetimes. Here we construct such a model in BD cosmology.

II. BACKGROUND UNIVERSE IN BRANS-DICKE THEORY

BD theory is described by the action,

$$S = \int d^4x\sqrt{-g}\left[\frac{\phi}{16\pi}\mathcal{R} - \frac{\omega}{16\pi\phi}(\nabla_\mu\phi)^2 + \mathcal{L}_m\right], \tag{2.1}$$

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where ϕ is BD field, ω is BD parameter, and $\mathcal{L}_m$ is the matter Lagrangian. The variations of equation (2.1) with respect to $g_{\mu\nu}$ and ϕ yield the field equations:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi}\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right] + \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi, \tag{2.2}$$

$$\Box\phi = \frac{8\pi}{2\omega+3}\mathrm{Tr}T. \tag{2.3}$$

As a background universe, we assume the FRW spacetime:

$$ds^2 = -dt^2 + a(t)^2\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right\}, \tag{2.4}$$

where $k = 0, \pm 1$ determines the spatial curvature. As an energy-momentum tensor, we introduce a dust fluid:

$$T_{\mu\nu} = \rho u_\mu u_\nu, \tag{2.5}$$

where ρ and u_μ are the density and the four-velocity of dust, respectively. The field equations (2.2) and (2.3) reduce to the following equations for the background universe:

$$H^2 + \frac{k}{a^2} = \frac{8\pi\rho}{3\phi} - H\frac{\dot{\phi}}{\phi} + \frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2, \tag{2.6}$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{8\pi\rho}{2\omega+3}, \tag{2.7}$$

where an overdot denotes d/dt and $H \equiv \dot{a}/a$ is the Hubble parameter.

In the flat space case, with $k = 0$, we know the general analytic solution for a dust universe [8]:

$$a(t) = a_0(t-t_+)^{\lambda_+}(t-t_-)^{\lambda_-}, \quad \phi(t) = \phi_0(t-t_+)^{\kappa_+}(t-t_-)^{\kappa_-}, \tag{2.8}$$

with constants $\lambda_\pm$ and $\kappa_\pm$ defined by

$$\lambda_\pm = \frac{\omega+1\pm\sqrt{1+2\omega/3}}{3\omega+4}, \quad \kappa_\pm = \frac{1\pm 3\sqrt{1+2\omega/3}}{3\omega+4}, \tag{2.9}$$

and a_0, ϕ_0 and $t_\pm$ are arbitrary constants.

If we take $t_+ = t_-$, the general solution (2.8) reduces to the special power-law solution, (used for example by Saida and Soda [6]):

$$a(t) = a_0(t-t_0)^{\frac{2\omega+2}{3\omega+4}}, \quad \phi(t) = \phi_0(t-t_0)^{\frac{2}{3\omega+4}}. \tag{2.10}$$

where t_0 may be set to zero. In the limit of $t \to \infty$, the general solution (2.8) converges to the special power-law solution (2.10). If the present cosmic age t is large enough, observations cannot constrain the relation between t_+ and t_-. Therefore, we keep $t_+ - t_-$ a free parameter.

In vacuum case ($\rho = 0$) there are analytic solutions for all k. The vacuum solution for $k = 0$ is expressed as [9]

$$a(t) = a_0 t^{\frac{1}{3(1+\alpha)}}, \quad \phi = \phi_0 t^{\frac{\alpha}{1+\alpha}}, \tag{2.11}$$

with

$$\alpha = \frac{1\pm\sqrt{1+2\omega/3}}{\omega}, \tag{2.12}$$

where we have omitted the arbitrary constant t_0 by fixing the origin of the time coordinate t. Introducing the conformal time $\eta = \int dt/a$, the vacuum solutions for $k = \pm 1$ are expressed as

$$k = +1 : a(\eta) = (\sin\eta)^{\frac{1-\lambda}{2}}(\cos\eta)^{\frac{1+\lambda}{2}}, \ \phi(\eta) = (\tan\eta)^\lambda, \tag{2.13}$$

$$k = -1 : a(\eta) = (\sinh\eta)^{\frac{1-\lambda}{2}}(\cosh\eta)^{\frac{1+\lambda}{2}}, \ \phi(\eta) = (\tanh\eta)^\lambda, \tag{2.14}$$

with

$$\lambda = \pm\frac{3}{3+2\omega}. \tag{2.15}$$

III. MODIFIED SWISS-CHEESE MODEL

Now we consider a model for a black hole embedded in the FRW universe. We replace a sphere in the FRW spacetime with a vacuum region which contains a black hole. Here "vacuum" means $T_{\mu\nu}=0$, and does not imply that $\mathcal{R}_{\mu\nu}=0$ due to the existence of BD field.

Extending Israel's junction conditions for a singular (or regular) hypersurface [10], Sakai and Maeda have studied bubble dynamics in the inflationary universe [11]. It was found that one can solve the equations of motion for the boundary without knowing the interior metric if the interior is vacuum, $T_{\mu\nu}=0$, or has only vacuum energy (a cosmological constant), $T_{\mu\nu}=-\rho g_{\mu\nu}$. Applying this method to the present model, we can determine the mass and the radius of a black hole without specifying the interior metric, as we shall show below.

Let us consider a spherical hypersurface Σ which divides a spacetime into two regions, V^+ (outside) and V^- (inside). We define a unit space-like vector, N_μ, which is orthogonal to Σ and points from V^- to V^+. In order to describe the behaviour of the boundary, we introduce a Gaussian normal coordinate system, $(n,x^i)=(n,\tau,\theta,\varphi)$, where τ is chosen to be the proper time on the boundary. Hereafter, we denote by $\Psi^\pm$ the value of any field variable Ψ defined on Σ by taking limits from $V^\pm$.

For the matter field, we consider a dust (or vacuum) medium for V^+ and vacuum for V^-:

$$T_{\mu\nu}{}^{+}=\rho u_\mu u_\nu,\quad T_{\mu\nu}{}^{-}=0. \tag{3.1}$$

Although we assume a smooth boundary at which there is no surface density, it is not obvious that this matching is possible at all times. Therefore, we introduce a surface energy-momentum tensor on the boundary surface,

$$S_{ij}\equiv\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}dn\;T_{ij}=\mathrm{diag}(-\sigma,\ \varpi,\ \varpi), \tag{3.2}$$

where σ and ϖ denote the surface energy-density and the surface pressure of Σ, respectively.

If we introduce the extrinsic curvature tensor of the world hypersurface Σ, $K_{ij}\equiv N_{i;j}$, we can write the junction conditions on Σ as [11]

$$[K_{ij}]^\pm=-\frac{4\pi}{\phi}\left(S_{ij}-\frac{\omega}{2\omega+3}\mathrm{Tr}S\gamma_{ij}\right), \tag{3.3}$$

$$-\;S_{i\ \ |j}^{j}=[T_i^n]^\pm, \tag{3.4}$$

$$K_{ij}^{+}S_i^j+\frac{2\pi}{\phi}\left\{S_j^iS_i^j-\frac{\omega}{2\omega+3}(\mathrm{Tr}S)^2\right\}=[T_n^n]^\pm, \tag{3.5}$$

where we have defined the jump in any quantity Ψ by the bracket $[\Psi]^\pm\equiv\Psi^+-\Psi^-$ and the three-dimensional covariant derivative by the vertical bar $|$. The junction condition for BD field is derived from equation (2.3) as

$$[\phi_{,n}]^\pm=-\frac{24\pi}{3+2\omega}\mathrm{Tr}S,\quad \phi^+=\phi^-, \tag{3.6}$$

which implies that ϕ is continuous at Σ and inhomogeneous in V_-.

The extrinsic curvature tensor of Σ in the homogeneous region V^+ is given by [11]

$$K_\tau^\tau=\gamma^3\frac{dv}{dt}+\gamma vH, \tag{3.7}$$

$$K_\theta^\theta=\frac{\gamma(1+vHR)}{R}=\frac{\epsilon}{R}\sqrt{1+\left(\frac{dR}{d\tau}\right)^2-R^2\left(H^2+\frac{k}{a^2}\right)}, \tag{3.8}$$

where

$$R=a(t)r|_\Sigma,\quad v\equiv a\frac{dr}{dt}\Big|_\Sigma,\quad \gamma\equiv\frac{\partial t}{\partial\tau}\Big|_\Sigma=\frac{1}{\sqrt{1-v^2}},\quad \text{and}\quad \epsilon\equiv\mathrm{sign}(K_\theta^\theta)=\mathrm{sign}\left(\frac{\partial R}{\partial n}\right). \tag{3.9}$$

From equations (3.1), (3.2), (3.4), (3.5), (3.7)-(3.9), we obtain the equations of motion:

$$\frac{dR}{dt} = \frac{dr}{d\chi}v + HR, \tag{3.10}$$

$$\gamma^3 \frac{dv}{dt} = -\gamma\left\{\left(1-2w\right)vH - \frac{2w}{R}\frac{dr}{d\chi}\right\} + \frac{2\pi\sigma}{\phi}\left\{1 + 4w + \frac{(1-2w)^2}{(2\omega+3)}\right\} - \frac{\gamma^2 v^2 \rho}{\sigma}, \tag{3.11}$$

$$\frac{d\sigma}{dt} = -\frac{2\sigma(1+w)}{R}\frac{dR}{dt} + \gamma v \rho, \tag{3.12}$$

where $w \equiv \varpi/\sigma$.

Once initial values of R, v, and σ are given, the equations of motion (3.10)-(3.12) determine their evolution. As discussed in [11], initial values should satisfy the angular component of (3.3),

$$\gamma(1+vHR) - \epsilon^- \sqrt{1 + \left(\frac{dR}{d\tau}\right)^2 - \frac{R_{MS}}{R}} = -\frac{8\pi\sigma R}{\phi}\left(\frac{\omega+1+w}{2\omega+3}\right), \tag{3.13}$$

where we have chosen $\epsilon^- = +1$. R_{MS} is defined by

$$R_{MS} \equiv R^-(1 - g^{\mu\nu}R^-_{,\mu}R^-_{,\nu}), \tag{3.14}$$

where R^- is defined as $R^- \equiv \sqrt{g_{\theta\theta}}$ at Σ on the V^-side. Since the Misner-Sharp mass is defined as [12]

$$M_{MS} \equiv \frac{R^-}{2G}(1 - g^{\mu\nu}R^-_{,\mu}R^-_{,\nu}) = \frac{R_{MS}}{2G}, \tag{3.15}$$

we call R_{MS} the "Misner-Sharp radius". Note that R_{MS} is a purely geometrical quantity and independent of theories of gravitation.

If we considered a spherical bubble in which there is no black hole (or singularity), we would have to solve the field equations with a regularity condition at the centre and the boundary condition (3.13), as done in [11]. However, because we are interested in black hole solutions, we do not have to take a central regularity condition into account. Thus, we can use equation (3.13) to determine R_{MS}.

At the initial time, we suppose $v = 0$ and $\sigma = \varpi = 0$, so R_{MS} is given by

$$R_{MS} = R^3\left(H^2 + \frac{k}{a^2}\right). \tag{3.16}$$

Let us now discuss whether v and σ remain zero during the ensuing evolution. Suppose $w = 0$, then the only nontrivial term in equation (3.11) is $\gamma^2 v^2 \rho/\sigma$. If v and σ evolved from zero, then equation (3.13) shows $vH \sim \sigma/\phi$, so that $\rho v^2/\sigma \sim \rho v/H\phi$. Therefore, equations (3.11) and (3.12) guarantee that, if $v = \sigma = 0$ at a certain time, $v = \sigma = 0$ at all time. Interestingly, this result is true only for the dust case, $\varpi/\sigma = 0$; otherwise the term $(2w/R)(dr/d\chi)$ in equation (3.11) would shift v from zero.

In the case of Schwarzschild spacetime, the Misner-Sharp radius coincides with the event horizon. Although this is not necessarily true for general spacetimes, we speculate that the Misner-Sharp radius is a well-defined measure of the size of a black hole. In the next section, we calculate the evolution of R_{MS} for black holes in several background cosmological models.

IV. EVOLUTION OF BLACK HOLES

The evolution of the Misner-Sharp radius for the $k = 0$ dust universe is given by equations (2.8) and (3.16),

$$R_{MS} = a_0^3 r_0^3 \left(\frac{\lambda_+}{t-t_+} + \frac{\lambda_-}{t-t_-}\right)^2 (t-t_+)^{3\lambda_+}(t-t_-)^{3\lambda_-}, \tag{4.1}$$

where r_0 is the comoving radius of the vacuum region. Equation (4.1) shows that the black hole size decreases with time.

If we define the black hole mass by

$$M_{MS} \equiv \frac{\phi R_{MS}}{2}, \tag{4.2}$$

then it coincides with the mass defined by Saida and Soda [6]. For the $k = 0$ dust universe, we obtain

$$M_{MS} = \frac{a_0^3 \phi_0}{2} \left(\frac{\lambda_+}{t - t_+} + \frac{\lambda_-}{t - t_-} \right)^2 (t - t_+)^{3\lambda_+ + \kappa_+} (t - t_-)^{3\lambda_- + \kappa_-}. \tag{4.3}$$

It is easy to see that equation (4.3) reduces to $M_{MS} =$ constant, if we choose $t_+ = t_-$, which is the same result as that found by Saida and Soda [6]. They also showed M_{MS} increases for $k = +1$ and decreases for $k = -1$, and concluded that the evolution of the mass depends qualitatively on the sign of the curvature of the universe. We should note, however, that their conclusion is true only for the special case $t_+ = t_-$, or equivalently, only for the asymptotic behavior of M_{MS} at $t \to \infty$. Our results give the general solution for all times.

Next, let us consider the scalar-field dominated (vacuum) universe. The time-dependent solutions for R_{MS} and M_{MS} in flat, open, and closed universes are given by

$$k = 0 : R_{MS} = \frac{a_0^3 r_0^3}{9(1+\alpha)^2} t^{\frac{-1-2\alpha}{1+\alpha}}, \tag{4.4}$$

$$M_{MS} = \frac{a_0^3 r_0^3 \phi_0}{18(1+\alpha)^2} t^{-1}, \tag{4.5}$$

$$k = +1 : R_{MS} = \frac{r_0^3}{2} (\cos\eta)^{-\frac{3+\lambda}{2}} (\sin\eta)^{-\frac{1+\lambda}{2}} (\cos 2\eta - \sin 2\eta - \lambda), \tag{4.6}$$

$$M_{MS} = \frac{r_0^3}{4} (\tan\eta)^{\lambda} (\cos\eta)^{-\frac{3+\lambda}{2}} (\sin\eta)^{-\frac{1+\lambda}{2}} (\cos 2\eta - \sin 2\eta - \lambda), \tag{4.7}$$

$$k = -1 : R_{MS} = \frac{r_0^3}{2} (\cosh\eta)^{-\frac{3+\lambda}{2}} (\sinh\eta)^{-\frac{1+\lambda}{2}} (\cosh 2\eta - \sinh 2\eta - \lambda), \tag{4.8}$$

$$M_{MS} = \frac{r_0^3}{4} (\tanh\eta)^{\lambda} (\cosh\eta)^{-\frac{3+\lambda}{2}} (\sinh\eta)^{-\frac{1+\lambda}{2}} (\cosh 2\eta - \sinh 2\eta - \lambda). \tag{4.9}$$

If we take $\omega > 500$, as applies to the universe today, then both $\alpha \sim O(\omega^{-\frac{1}{2}})$ and $\lambda \sim O(\omega^{-1})$ will be negligible. We see that R_{MS} and M_{MS} both decrease with increasing time, except in the contracting phase of the $k = +1$ universe. In a more general scalar-tensor theory, with non-constant $\omega(\phi)$ we would expect similar effects to arise and it would be possible for large changes in $\omega(\phi)$ to occur in the very early universe despite very slow evolution towards a very large value of $\omega > 1000$ today. However, such a calculation should be performed for a black hole in radiation and dust-dominated universes.

V. DISCUSSION

We have constructed a modified "Swiss cheese" model in Brans-Dicke theory and discussed the evolution of black holes for dust and vacuum universes. We defined the size of a black hole by the Misner-Sharp mass R_{MS} , and found that this size always decreases as long as the universe is in an expanding phase. From an observational point of view, R_{MS} is a well-defined quantity, which reveals the gravitational effects of the black hole on a distant observer. Although we have not specified the metric around the black hole, the black hole mass and radius that we have obtained coincide with those used by Saida and Soda [6], who assumed a Schwarzschild-like metric. This shows that their ansatz of the Schwarzschild-like metric does not introduce a specialization of the problem. We have not investigated the behavior of the interior vacuum region. However, the behaviour of a black hole in a vacuum spacetime was investigated by Scheel et al. [3] and Jacobson [5]. Their analyses suggest that the scalar field increases at the horizon and hence the horizon area decreases with time as the background universe expands. Although Jacobson's particular solution of the scalar wave equation, with very fast variation of $\phi(t, \infty) \propto t$ in the background universe, look unrealistic, his conclusion may be unchanged for a more realistic boundary cosmological condition because homogenisation of BD field occurs more effectively for slower time-variation of $\phi(t, \infty)$ [13].

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